DYNAMICS OF BELIEFS AND LEARNING UNDER $\mathbf{a}_L$-PROCESSES – THE HETEROGENEOUS CASE

CARL CHIARELLA AND XUE-ZHONG HE

School of Finance and Economics

University of Technology, Sydney

PO Box 123 Broadway

NSW 2007, Australia

ABSTRACT. This paper studies a class of models in which agents' expectations influence the actual dynamics while the expectations themselves are the outcome of some recursive processes with bounded memory. Under the assumptions of heterogeneous expectations (or beliefs) and that the agents update their expectations by recursive L- and general $\mathbf{a}_L$-processes, the dynamics of the resulting expectations and recursive schemes are analyzed. It is shown that the dynamics of the system, including stability, instability and bifurcation, are affected differently by the recursive processes. The cobweb model with a simple heterogeneous expectation scheme is employed as an example to illustrate the stability results, the various types of bifurcations and the routes to complicated price dynamics. In particular, the double edged effect of heterogeneity on the dynamics of the model is demonstrated.

Keywords: Heterogeneous beliefs, recursive L-process, general $\mathbf{a}_L$-process, stability, instability, bifurcation, cobweb model.

We are indebted to Cars Hommes, Laura Gardini and the anonymous referees for a number of worthwhile suggestions that have considerably improved the paper. The usual caveat applies. Financial support from Australian Research Council grant A79802872 is greatly acknowledged.

1. INTRODUCTION

Many dynamic economic models form an expectations feedback system. Expectations affect actual outcomes, actual outcomes affect expectations through learning, and so on. Properties of various learning processes under homogeneous expectations have been studied extensively (see, for example, Balasko and Royer (1996), Bray (1983), Evans and Honkapohja (1999), Evans and Ramey (1992), Lucas (1978) and Marcet and Sargent (1989)). In his survey paper, Grandmont (1998) considers stability and convergence to self-fulfilling expectations in large socioeconomic systems and suggests a kind of general *'Uncertainty Principle' – Learning is bound to generate local instability of self-fulfilling expectations, if the influence of expectations on the dynamics is significant.* When learning processes are involved, as pointed out by Balasko and Royer (1996), *'the properties of the (Walrasian) equilibrium with respect to the convergence of least squares learning processes and, more generally, of recursive processes have hardly been studied'*.

Research into the dynamics of financial asset prices resulting from the interaction of heterogeneous agents having different expectations about the future evolution of prices has flourished in recent years, e.g. Brock and Hommes (1997*a*), (1997*b*), (1998), Bullard (1994), Bullard and Duffy (1999), Chiarella and He (2001*b*), Day and Huang (1990), Franke and Nesemann (1999), Franke and Sethi (1998), Hommes (1998), Levy and Levy (1996) and Lux (1995), (1997), (1998). As indicated by Levy and Levy (1996), *'heterogeneous expectations appear to play a crucial role in risky asset price determination. When homogeneous expectations are assumed, unacceptable market inefficiencies are observed. The introduction of even a small degree of diversity of expectations changes the dynamics dramatically, and the result is a much more realistic market'*.

These observations lead to the following questions: *What are the effects of heterogeneous expectations on the dynamics of the state variables of economic systems? Are the homogeneous expectations models "approximately" correct? When heterogeneous* *expectations are involved, do learning processes affect the dynamics of economic models differently?* These questions have been tackled recently from different points of view for various economic models (e.g. Balasko and Royer (1996), Grandmont (1998), Brock and Hommes (1997*a*), (1997*b*), (1998), Chiarella and He (2001*b*), Franke and Nesemann (1999), Franke and Sethi (1998), Hommes (1998), Levy and Levy (1996)) but, at this stage, no satisfactory general theory has emerged.

This paper is largely motivated by the above observations and questions, but concentrates on the question as to how the recursive L and the general $\mathbf{a}_L$ learning process (defined in the following section) affect the dynamics, in particular the stability and bifurcation, of (Walrasian) equilibria of economic models with heterogeneous beliefs. In this paper, the term learning is being used in a very particular, and perhaps restricted sense. It refers to a situation in which agents adopt a rule to come up with an expectation of next period's price. A broader use of the term learning would envisage a situation in which agents are able to switch strategies in light of prediction errors, for example as in Brock and Hommes (1997*b*). This paper considers a deterministic (nonlinear) framework and focuses on an extremely simple case, in which the state of the system is completely described at every date by a single real number x_t. Depending upon the context, the state variable x may stand for a price, a rate of inflation, a real rate of interest etc. Traders plan one period ahead. To abstract from all forms of uncertainty, the traders' expectations or forecasts follow finite general $\mathbf{a}_L$- or recursive L-processes.

Under the homogeneous expectation assumption, Chiarella and He (2001*a*) provide an explicit study of how the local stability of the fixed equilibrium and types of bifurcation (to complicated dynamics) are affected by recursive L- and $\mathbf{a}_L$-process. In particular, their study shows that, when agents follow homogeneous recursive L-process, the stability and bifurcation of the fixed equilibrium can be completely characterized by the parameters of the system and the lag length L of the learning process. The fixed equilibrium becomes unstable through either a saddle-node or a Neimark-Hopf (or secondary Hopf) bifurcation, leading to complicated dynamics. However, when agents follow a homogeneous $\mathbf{a}_L$-process, the dynamics of the system depends on both lag length L and weight vector $\mathbf{a}$ and more complicated dynamics can arise.

This paper generalizes the recent study on the dynamics of homogeneous expectations in Chiarella and He (2001a) and concentrates on how the dynamics, including the stability and bifurcation of the system, is affected by the introduction of heterogeneous recursive L- and general $\mathbf{a}_L$-processes. In particular the paper shows how the dynamics of such processes are affected by agents' extrapolation rates, lag lengths used to form expectations and the way in which past information is weighted. The paper is structured as follows. The heterogeneous process is introduced in Section 2. The dynamics (including stability and bifurcation) of the heterogeneous recursive L- and the general $\mathbf{a}_L$-processes is then discussed in Sections 3 and 4. In Section 5, the theoretical results of the earlier sections are used to undertake an extensive bifurcation analysis of version of the nonlinear adaptive beliefs cobweb model established by Brock and Hommes (1997b) under heterogeneous expectations. Section 6 concludes. The appendix contains the technical details of the various propositions and bifurcation analyses given in the paper.

2. Heterogeneous Beliefs and Learning

To introduce the model with heterogeneous beliefs, it is assumed that there are m different types of traders, indexed by $j = 1, ..., m$, and the j-th type of trader's forecast at date t about the future state is denoted by ${}_{t-1}x^e_{j,t+1}$ $(j = 1, 2, \cdots, m)$. Assuming x_t is not included in the information set, then the temporary equilibrium relation becomes

$$T(x_t,\ {}_{t-1}x^e_{1,t+1}, \cdots,\ {}_{t-1}x^e_{m,t+1}) = 0. \tag{2.1}$$

Each type of trader's learning process is summarized by a continuously differentiable expectation function (formed from the past L_j state variables x_{t-k} for $k = 1, \cdots, L_j$)

$${}_{t-1}x^e_{j,t+1} = \psi_j(x_{t-1}, \cdots, x_{t-L_j}) \qquad (j = 1, \cdots, m). \tag{2.2}$$

Assume that there exists x^* such that $T(x^*, y_1^*, y_2^*, \cdots, y_m^*) = 0$, where $y_j^* = \psi_j(x^*, x^*, \cdots, x^*)$ for $j = 1, 2, \cdots, m$. Then x^* is a fixed (Walrasian) equilibrium of (2.1). It is also assumed that T is continuously differentiable near the fixed equilibrium x^*. Near the (Walrasian) equilibrium x^* the dynamics are characterized by a linear system. Denote

$$B = \frac{\partial T(x, y_1, \cdots, y_m)}{\partial x}\bigg|_{(x^*, y_1^*, \cdots, y_m^*)}, \quad C_j = \frac{\partial T(x, y_1, \cdots, y_m)}{\partial y_j}\bigg|_{(x^*, y_1^*, \cdots, y_m^*)}$$

and assume $B, C_j \neq 0 (j = 1, \cdots, m)$.

For $j = 1, \cdots, m$, consider L_j real numbers $a_{jk} \geq 0$[1] satisfying $\sum_{k=1}^{L_j} a_{jk} = 1$. The general heterogeneous recursive $\mathbf{a}_L$- and the recursive L-processes are defined as follows.

Definition 2.1. *The* **general heterogeneous recursive (finite) $\mathbf{a}_L$-process** *is defined by (2.1), (2.2) and the expectation function*

$$\psi_j(x_1, \cdots, x_{L_j}) = g_j(a_{j1}x_1 + \cdots + a_{jL_j}x_{L_j}), \quad 0 \leq a_{jk} \leq 1, \quad \sum_{k=1}^{L_j} a_{jk} = 1 \quad (2.3)$$

where g_j $(j = 1, \cdots, m)$ are some (locally near x^*) continuously differentiable functions. The **heterogeneous recursive L-process** *is simply the $\mathbf{a}_L$-process with weights satisfying $a_{jk} = 1/L_j$ for $j = 1, \cdots, m$ and $k = 1, \cdots, L_j$.*

The linearized equation (near x^*) of the general heterogeneous recursive (finite) $\mathbf{a}_L$-process (2.1)–(2.3) is given by

$$B(x_t - x^*) + \sum_{j=1}^{m} C_j g_{jo} \sum_{k=1}^{L_j} a_{jk}(x_{t-k} - x^*) = 0, \quad (2.4)$$

[1] Here a_j are treated as the weights (or probabilities) of the past states and therefore are assumed to be nonnegative.

where $g_{jo} = g'_j(x^*)$[2]. Replacing $x_t - x^*$ by x_t, then the stability of the steady-state x^* of (2.4) is equivalent to the stability of the zero solution of the difference equation

$$x_t + \sum_{j=1}^{m} \alpha_j \sum_{k=1}^{L_j} a_{jk} x_{t-k} = 0, \qquad \alpha_j = C_j g_{jo}/B. \tag{2.5}$$

We stress that each α_j may be independently positive or negative (or zero) depending on the signs of B, C_j and g_{jo}. Let $L = \max_{1\le j\le m}\{L_j\}$ and define $a_{jk} = 0$ for $j = 1, \cdots, m$ and $k = L_j + 1, \cdots, L$. Then equation (2.5) can be written as

$$x_t + \sum_{k=1}^{L} \left(\sum_{j=1}^{m} \alpha_j a_{jk} \right) x_{t-k} = 0. \tag{2.6}$$

Therefore the local stability of the general heterogeneous recursive (finite) $\mathbf{a}_L$-process is generically governed by the eigenvalues of the characteristic polynomial of (2.6):

$$\Gamma(\lambda) \equiv \lambda^L + \sum_{k=1}^{L} \left(\sum_{j=1}^{m} \alpha_j a_{jk} \right) \lambda^{L-k} = 0. \tag{2.7}$$

Following Grandmont (1998), the coefficients of equation (2.6) can be interpreted in the following way. Suppose the expectation coefficients in aggregate $C \equiv \sum_{j=1}^{m} C_j \neq 0$. Let $\gamma_j = C_j/C$ be the relative local contribution of the jth expectation and $g \equiv \sum_{j=1}^{m} \gamma_j g_j$ be the weighted average expectation. In models of asset prices, the coefficients g_{jo} may characterize the extrapolation rate and γ_j may relate to the fraction of the j-th type of traders who follow the $j-$th expectation. Then $\sum_{j=1}^{m} \alpha_j a_{jk} = (C/B)\sum_{j=1}^{m} \gamma_j g_{jo} a_{jk}$ is the derivative of the average expectation g evaluated at x^*, weighted by the weights of all $\mathbf{a}_L$-processes associated with state variable x_{t-j}. In particular, if all the traders follow a homogeneous belief, then $g_j = g$, $a_{jk} = a_j$ for $j = 1, \cdots, m$ and hence $\sum_{j=1}^{m} \alpha_j a_{jk} = (C g_o/B) a_j$, which reduces the equation (2.6) to

$$\Gamma(\lambda) \equiv \lambda^L + \alpha \sum_{j=1}^{L} a_j \lambda^{L-j} = 0, \qquad \alpha = C g_o/B \tag{2.8}$$

[2]The slope of the expectation function g_j evaluated at x^* can be used to characterize the extrapolation rate of the j-th type of trader. Broadly speaking its sign indicates whether trader j is a trend chaser ($g'_j > 0$) or contrarian ($g'_j < 0$).

that Chiarella and He (2001a) found to be the characteristic polynomial for the homogeneous $\mathbf{a}_L$ process. Therefore, the discussion in this paper is a natural generalization of that in Chiarella and He (2001a).

The analysis in section 3 considers first the case of the heterogeneous recursive L-process and then section 4 moves to the case of the general heterogeneous $\mathbf{a}_L$-process.

3. Heterogeneous Recursive L-Processes

As a special case of the general heterogeneous $\mathbf{a}_L$-process, the heterogeneous recursive L-process is defined by taking $a_{jk} = 1/L_j$ for $j = 1, \cdots, m$ and $k = 1, \cdots, L_j$. The following lemma will be used in our discussion.

Lemma 1. *(Chiarella and He (2000)) Let*

$$Q_L(\lambda) \equiv \lambda^L + \gamma\lambda^{L-1} + \gamma\lambda^{L-2} + \cdots + \gamma\lambda + \gamma. \tag{3.1}$$

Then zeros of $Q_L(\lambda)$ lie inside the unit circle if and only if $-\frac{1}{L} < \gamma < 1$.

In the general situation of the heterogeneous recursive L-process, different types of agents may use different lag lengths in their recursive L-processes. In this case, to obtain necessary and sufficient conditions for local asymptotic stability (**LAS**) seems an intractable problem. To gain some insights into the dynamics in this situation, some simple cases are considered in the following analysis.

3.1. **The Reduced Recursive Homogeneous Case: $L_j = L$.** Consider first the case when agents use the same lag length, that is $L_j = L$ and hence $a_{jk} = 1/L$ for $j = 1, \cdots, m, k = 1, 2, \cdots, L$, but with a different form of extrapolation function g_j. Then the corresponding characteristic polynomial $\Gamma(\lambda)$ has the form of (3.1) with $\gamma = \sum_{j=1}^{m} \alpha_j/L$. Applying Lemma 1 and the bifurcation results of Chiarella and He (2001a) leads to the following result on the LAS of the heterogeneous recursive L-process.

Proposition 3.1. *Let $L_j = L$ for $j = 1, \cdots, m$. Then the (Walrasian) equilibrium x^* of the heterogeneous recursive L-process is locally asymptotically stable (LAS) if* [3]

$$-1 < \alpha_o \equiv \sum_{j=1}^{m} \alpha_j < L. \tag{3.2}$$

Furthermore at $\alpha_o = -1$ a saddle bifurcation occurs and at $\alpha_o = L$ a Neimark-Hopf (or $1 : (L+1)$ periodic resonance[4]*) bifurcation occurs for $L \geq 1$.*

Denote by $D'_L(\alpha_o) = \{\alpha_o : -1 < \alpha_o < L\}$ the stability region for the parameter α_o corresponding to the heterogeneous recursive L-process. Proposition 3.1 indicates that, when agents follow the heterogeneous recursive L-process using the same lag length, the LAS of the equilibrium x^* is completely characterized by $D'_L(\alpha_o)$. Obviously, $D'_L(\alpha_o) \subset D'_{L'}(\alpha_o)$ for $L < L'$, indicating that increase of the lag length widens the stability region of the equilibrium.

Using the notations of section 2, condition (3.2) can be rewritten as

$$-1 < \alpha_o \equiv \frac{C}{B}\bar{g}_o < L \qquad \text{with} \qquad \bar{g}_o = \sum_{j=1}^{m} \gamma_j g_{jo}. \tag{3.3}$$

In the case of homogeneous beliefs, $g_j = g$ and hence $g_{jo} = g_o$ for all $j = 1, \cdots, m$, the condition (3.3) becomes the condition for the homogeneous recursive model derived in Chiarella and He (2001*a*). The parameter α_o in (3.3) can be viewed as an aggregate extrapolation rate for the heterogeneous recursive model that brings some new features. For instance, although each individual's expectation rule may involve

[3] Condition (3.2) is a necessary and sufficient condition for the linearized system (at the fixed point x^*) to be LAS. However, for nonlinear system, this condition is not necessary because the fixed point may be LAS also when some of the eigenvalues lie on the unit circle, this may occur in the pitchfork bifurcation. We are indebted to Laura Gardini for drawing this point to our attention.

[4] For a map in $\mathbb{R}^2$, when all the eigenvalues are on the unit circle, there is no *"(strong) resonance"* if there is an eigenvalue, say $\bar{\lambda}$, satisfying $\bar{\lambda}^q \neq 1$ for $q = 1, 2, 3, 4$. Otherwise, we say the map has a $1 : q$ *(strong) resonance* ($q = 1, 2, 3, 4$). When the nonresonance condition is satisfied, for a $\mathbb{R}^2$ map depending on one parameter, as the eigenvalues of the fixed equilibrium move off the unit circle, there appears a closed invariant curve — all the iterates of any point on the curve remain on the curve — encircling the fixed point. Such a bifurcation corresponds to the Poincare-Andronov-Hopf bifurcation, see also Hale and Kocak (1991) for more discussion. When the nonresonance condition is not satisfied, a $1 : q$ (strong) resonance bifurcation in $\mathbb{R}^2$ can generate a two orbits of period q — one orbit is a sink and the other is a saddle.

significantly unstable elements (for example, large extrapolation rates g_{io}), these elements may be *"balanced"* in the aggregate, e.g. $\bar{g}_o$ may be small, and the actual dynamics with the heterogeneous recursive learning process may thus be LAS, provided there is sufficient balance[5] in the heterogeneity. In such cases, learning can stabilize an otherwise unstable dynamics (e.g. Franke and Sethi (1998)). On the other hand, only a small group of traders with expectation functions involving significantly divergent elements (say, for example, there exists a $k : 1 \leq k \leq m$ such that (3.3) holds for $\bar{g}_o \equiv \bar{g}_k = \sum_{j \neq k} \gamma_j g_{jo}$, but not for $\bar{g}_o = \sum_{j=1}^{m} \gamma_j g_{jo}$) can in fact destabilize the whole system (e.g. Grandmont (1998)[6]) and this corresponds to a popular view (particularly in asset price models, e.g. Brock and Hommes (1997*a*), (1997*b*), (1998), Hommes (1998)) that heterogeneous beliefs are a source of instability in the market and may lead to periodic or even chaotic fluctuations in prices. The forgoing discussion demonstrates that in actual fact the effect of heterogeneity on the stability of economic dynamic models is a double edged one.

3.2. **The Case of Two Beliefs.** Next consider the case when $m = 2$ and $L_1 < L_2$ for which one is able to obtain some sufficient conditions for LAS (see Appendix A.2 for the proof).

Proposition 3.2. *Assume $m = 2$ and $1 \leq L_1 < L_2$. If $\alpha_j = C_j g_{io}/B$ $(j = 1, 2)$ satisfy*

$$L_1 \left| \frac{\alpha_1}{L_1} + \frac{\alpha_2}{L_2} \right| + \frac{L_2 - L_1}{L_2} |\alpha_2| < 1, \tag{3.4}$$

then the (Walrasian) equilibrium x^* of the heterogeneous recursive L-process is LAS.

From Proposition 3.2, one can easily derive the following sufficient condition which is independent of L_j $(j = 1, 2)$.

Corollary 1. *Assume $m = 2$ and $1 \leq L_1 < L_2$. If $|\alpha_1| + |\alpha_2| < 1$, then the (Walrasian) equilibrium x^* of the heterogeneous recursive L-process is LAS.*

[5]The term balance here refers to offsetting values of α_j's such that $\alpha = \alpha_1 + \cdots + \alpha_m$ remains in the local stability region $D'_L(\alpha_o) = \{\alpha_o : -1 < \alpha_o < L\}$.

[6]The results developed in this paper are quite different from the results in Grandmont (1998), which relate the eigenvalues of the actual system to the perfect foresight eigenvalues.

When different types of traders use a differing number of the observations (to form their expectations) in the recursive L-process, one would expect to see an asymmetric effect with respect to different lag length. In particular, one might expect that, when one group of traders is willing to use more past observations to learn the equilibrium, this group would be able to extrapolate[7] over a wider range of rates. However, the following example indicates that this expectation is not always borne out.

Consider the case of $L_1, L_2 = 1, 2$ and let $D'_{ij}(\vec{\alpha})$ be the stability region of the fixed equilibrium in terms of $\vec{\alpha} = (\alpha_1, \alpha_2)$ for $(L_1, L_2) = (i, j)$. Then, for $(L_1, L_2) = (1, 2)$, the corresponding characteristic polynomials is $\Gamma_{1,2}(\lambda) \equiv \lambda^2 + (\alpha_1 + \alpha_2/2)\lambda + \alpha_2/2 = 0$. Applying Proposition 3.1 and Lemma 2 in Appendix A.3, one can obtain $D'_{12}(\vec{\alpha}) = \{(\alpha_1, \alpha_2) : \alpha_2 < 2, -[1 + \alpha_2] < \alpha_1 < 1\}$.

The stability regions $D'_{22}(\vec{\alpha}) = \{(\alpha_1, \alpha_2) : -1 < \alpha_1 + \alpha_2 < 2\}$ and $D'_{12}(\vec{\alpha})$ are plotted in the (α_1, α_2) plane in Fig.1. One can see that D'_{12} is the bounded triangular region, while $D'_{11}(\vec{\alpha}) = \{(\alpha_1, \alpha_2) : -1 < \alpha_1 + \alpha_2 < 1\}$ and D'_{22} are unbounded strips.

- When $\alpha_1 \in (0, 1)$, increasing $L_2 = 1$ to $L_2 = 2$ extends the stability region for $L_1 = L_2 = 1$ to the triangular region: $\{(\alpha_1, \alpha_2) : \alpha_1 < 1, \alpha_2 < 2, \alpha_1 + \alpha_2 \geq 1\}$. Therefore, increasing lag length L_2 increases the stability range of the parameter α_2 (which allow the type 2 traders to extrapolate over a wide range of rates).
- When $\alpha_1 \notin (0, 1)$, increasing $L_2 = 1$ to $L_2 = 2$ destabilizes an otherwise stable equilibrium x^* and leads the unbounded LAS strip region: $\{(\alpha_1, \alpha_2) : \alpha_1 \leq 0, -1 < \alpha_1 + \alpha_2 < 1\}$ for $L_2 = 1$ to a bounded triangular region: $\{(\alpha_1, \alpha_2) : \alpha_1 < 0, \alpha_2 < 2, \alpha_1 + \alpha_2 > -1\}$. In such a case, increasing the lag length L_2 reduces the stability range of the parameter α_2. A similar observation can be made when comparing the stability regions $D'_{12}(\vec{\alpha})$ with $D'_{22}(\vec{\alpha})$.
- When both L_1 and L_2 are the same, the stability regions are the unbounded strips, while any break of such symmetry (in terms of the lag lengths) leads to bounded triangular stability regions.

[7] Recall that agent j's extrapolation rate is captured in the coefficient α_j introduced at equation (2.5).

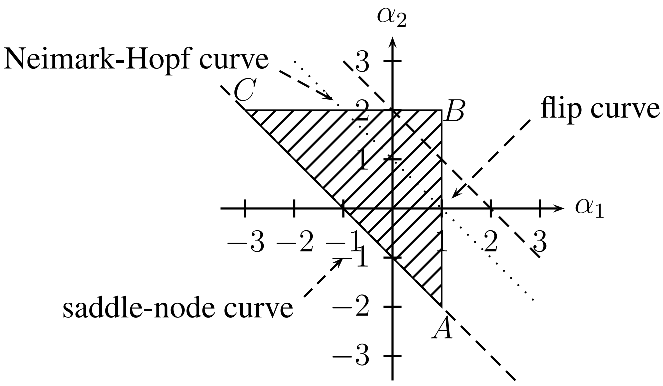


FIGURE 1. Local stability regions of the reduced homogeneous recursive processes $D'_{11}(\vec{\alpha})$, $D'_{22}(\vec{\alpha})$ and the heterogeneous recursive 2-process $D'_{12}(\vec{\alpha})$. $D'_{11}(\vec{\alpha})$ ($D'_{22}(\vec{\alpha})$) is the unbounded strip between the saddle-node boundary $\alpha_1 + \alpha_2 = -1$ and the Neimark-Hopf boundary $\alpha_1 + \alpha_2 = 1$ ($\alpha_1 + \alpha_2 = 2$), $D'_{12}(\vec{\alpha})$is the hatched triangle.

The stability regions $D'_{11}(\vec{\alpha}), D'_{22}(\vec{\alpha})$ and $D'_{12}(\vec{\alpha})$ indicated in Fig. 1 show that the fixed equilibrium is LAS in the triangular region ABC. However, when the parameter $\vec{\alpha} = (\alpha_1, \alpha_2)$ moves out of the triangular region, the fixed equilibrium becomes unstable and can lead to various types of bifurcation.

- Along CA, the two eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \alpha_2/2$ with $|\lambda_2| < 1$ and, according to Kuznetsov (1995) (Chapters 4 and 9), a saddle-node bifurcation appears. Such a curve (with one of the eigenvalues equal to 1) is called a *saddle-node (or divergent) curve* and the corresponding boundary of the stability region is called a *saddle-node boundary*.
- Along AB, the two eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -\alpha_2/2$ with $|\lambda_2| < 1$, such a bifurcation, according to Kuznetsov (1995), is called a *flip (or period doubling)* bifurcation. Such a curve (with one of the eigenvalues is equal to -1) is called a *flip curve* and the corresponding boundary of the stability region is called a *flip boundary*.
- Along BC, the eigenvalues satisfy $\lambda_j \in \mathbb{C}, |\lambda_j| = 1$ for $j = 1, 2$ and this corresponds to a *Neimark-Hopf* bifurcation. In fact, along BC, $\lambda_{1,2} = \cos(2\pi w) \pm i \sin(2\pi w)$ for $w \in \mathbb{R}$. Let $\rho = 2\cos(2\pi w)$, then the character of bifurcation is determined by the value w, which in turn is determined by the value ρ.

It follows from $\lambda_1\lambda_2 = \alpha_2/2 = 1$ and $\lambda_1 + \lambda_2 = \rho = -(\alpha_1 + \alpha_2/2)$ that $\alpha_1 = -(\rho+1)$ and $\alpha_2 = 2$ for the case of $(L_1, L_2) = (1, 2)$. Also $\alpha_1 \in [-3, 1]$ implies $\rho \in [-2, 2]$. This indicates that, along BC, the eigenvalues can have values of w satisfying $\rho = 2\cos(2\pi w) \in [-2, 2]$, that is, w can take any real value. If w is a rational fraction $w = p/q$, there exists a $p : q$ periodic resonance bifurcation. Table 4 in Appendix A.4 lists a $p : q$ resonance bifurcations and corresponding values of $\rho = 2\cos(2\pi p/q)$ for $q < 12$. Checking with Table 4 and using $\alpha_1 = -(\rho + 1)$, one can find a $p : q$ resonance bifurcation with the corresponding value of α_1 in Table 1. For example, $\alpha_1 = -2$ corresponds to periodic 6 bifurcation; for $\alpha_1 = -1$, there exist period 4 bifurcations starting the Feigenbaun period doubling route to chaos; for $\alpha_1 = 0$, there exist period 3 bifurcations; for $\alpha_1 = 1$, there exist period 2 bifurcations. If w is irrational, quasi-periodic orbits can be bifurcated. Such a curve (with resonance and quasi-periodic orbits bifurcations) is called a *Neimark-Hopf curve* and the corresponding boundary of the stability region is called a *Neimark-Hopf boundary*. Therefore, along the Neimark-Hopf boundary, the system can bifurcate to resonances and quasi-periodic orbits, which in turn lead to different routes to complicated dynamics, see Kuznetsov (1995) (Chapter 9) for more detailed discussion.

q	p	α_1		q	p	α_1
2	1	1		5	1,4	-1.61803
3	1	0			2, 3	0.61803
4	1	-1		6	1,5	-2

TABLE 1. Some $p : q$ resonance and the corresponding values α_1

To see how the bifurcation curves and types of bifurcation are changed for various lag lengths, consider two cases: $(L_1, L_2) = (1, 3)$ and $(2, 3)$. The local stability regions $D'_{13}(\vec{\alpha})$ and $D'_{23}(\vec{\alpha})$ are plotted in Fig.2 with flip, saddle-node and Neimark-Hopf boundaries as indicated (a detailed discussion of these two cases is contained in Appendix A.5).

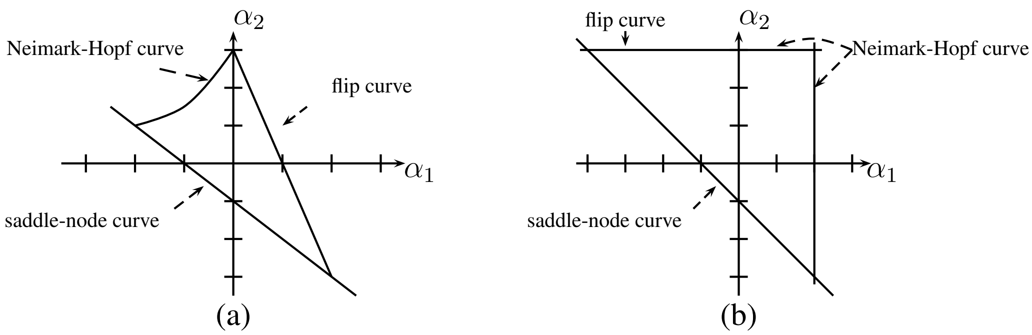


FIGURE 2. Local stability region: (a) $D'_{13}(\vec{\alpha})$ with saddle-node boundary $\alpha_1 + \alpha_2 = -1$, flip boundary $\alpha_1 + \alpha_2/3 = 1$ and Neimark-Hopf boundary $\alpha_2(1-\alpha_1) = 3$; (b) $D'_{23}(\vec{\alpha})$ with saddle-node boundary $\alpha_1 + \alpha_2 = -1$, flip boundary $\alpha_2 = 3$ and Neimark-Hopf boundaries $(F1) : \alpha_1 = 2, \alpha_2 \in [-3, 3]$ and $(F2) : \alpha_2 = 3, \alpha_1 \in [-4, 2]$.

In all these cases, along the Neimark-Hopf boundary, the character of the bifurcations is determined by the eigenvalues $\lambda_{1,2} = \exp(\pm 2\pi w i)$, which in turn is determined by the value of ρ. Table 2 summarize the region of ρ for heterogeneous least squares process with different lag length. One can see that the lags have different influence on the region of ρ, and hence on the types of bifurcation along the Neimark-Hopf boundary.

(L_1, L_2)	Region of ρ		(L_1, L_2)	Region of ρ
(1, 1)	-2		(1, 2)	[-2, 2]
(2, 2)	-1		(1, 3)	[0, 2]
(3, 3)	0		(2, 3)	[-1, 2]

TABLE 2. The corresponding region of $\rho = 2\cos(2\pi w)$ for different lag lengths (L_1, L_2)

The main results of this section can be summarized as follows:

- For the reduced homogeneous recursive L-process, the stability can be characterized completely by the aggregated extrapolation rate $\alpha = \sum_{j=1}^{m} \alpha_j$, saddle-node and $1 : (L+1)$ periodic resonance are the only types of bifurcation generated from the instability of the fixed equilibrium.
- For the case of two heterogeneous beliefs with different lag lengths, the stability regions for $\vec{\alpha} = (\alpha_1, \alpha_2)$ are bounded by saddle-node, flip and Neimark-Hopf boundaries. In particular, along the Neimark-Hopf boundary, various

types of resonances, quasi-periodic orbits and routes to chaos can be observed for different lag lengths.

4. The General Heterogeneous Recursive $\mathbf{a}_L$ Process

For general heterogeneous recursive $\mathbf{a}_L$ processes, the local stability is determined by the zeros of the characteristic polynomial (2.7). Using Rouche's Theorem (see Appendix A.1), one can obtain the following sufficient conditions for the LAS of the equilibrium x^* (see Appendix A.6 for the proof).

Proposition 4.1. *For the general heterogeneous $\mathbf{a}_L$-process,*

- *the (Walrasian) equilibrium x^* is LAS if (i): $\sum_{k=1}^{L} \left| \sum_{j=1}^{m} \alpha_j a_{jk} \right| < 1$;*
- *the equilibrium x^* is unstable if there exists $k_o : 1 \leq k_o \leq L$ such that (ii): $\left| \sum_{j=1}^{m} \alpha_j a_{jk_o} \right| > 1 + \sum_{k=1, k\neq k_o}^{L} \left| \sum_{j=1}^{m} \alpha_j a_{jk} \right|$.*

Proposition 4.1 is a generalization of the corresponding result for the homogeneous $\mathbf{a}_L$-process in Chiarella and He (2001a). In the following, the stability of heterogeneous $\mathbf{a}_2$- and $\mathbf{a}_3$-processes is characterized first and a bifurcation analysis on the heterogeneous $\mathbf{a}_2$-process then follows.

4.1. Stability of $\mathbf{a}_2-$ and $\mathbf{a}_3$-Processes.

Applying Lemma 2, one obtains the following stability result on heterogeneous $\mathbf{a}_2$- and $\mathbf{a}_3$-processes.

Proposition 4.2. *Let $L = \max_{1\leq j\leq m}\{L_j\}$. Then the equilibrium x^* is LAS if*

$$\alpha = \sum_{j=1}^{m} \alpha_j > -1, c_1 - c_2 < 1, c_2 < 1 \tag{4.1}$$

for $L = 2$ and

$$\alpha = \sum_{j=1}^{m} \alpha_j > -1, 1 - c_1 + c_2 - c_3 > 0, 1 - c_2 + c_3(c_1 - c_3) > 0 \tag{4.2}$$

and $c_2 < 3$ for $L = 3$, where $c_k = \sum_{j=1}^{m} \alpha_j a_{jk}$ for $k = 1, 2, 3$.

Obviously, the stability of the fixed equilibrium is determined not only by the various lag lengths $L_j = 1, 2, 3$ for $j = 1, 2$, but also by the weight vector $\mathbf{a}_{L_j}$. A simple case of an $\mathbf{a}_2$ process with $m = 2$ is considered in the following so that various bifurcations generated from the $\mathbf{a}_2$ process can be analyzed in detail.

4.2. **Bifurcation Analysis of the Heterogeneous $\mathbf{a}_2$-Process with $m = 2$.** Let $D'_{jk}(\vec{\alpha}, \mathbf{a})$ be the stability region in terms of $\vec{\alpha} = (\alpha_1, \alpha_2)$ and the general heterogeneous $\mathbf{a}_2$-process with $m = 2, L_1 = j, L_2 = k$ for $j, k = 1, 2$. To see the dynamics of the $\mathbf{a}_2$ process, let $L_1 = L_2 = L = 2$. Let the corresponding $\mathbf{a}_2$ processes be $\{(a_{j1}, a_{j2}) : a_{jk} \geq 0, a_{j1}+a_{j2} = 1\}$ for $j = 1, 2$ and denote $v_1 = a_{12}$ and $v_2 = a_{22}$. Then $a_{11} = 1-v_1$ and $a_{21} = 1 - v_2$. Following Proposition 4.2, the stability region $D'_{22}(\vec{\alpha}, \mathbf{a})$ is defined by $D'_{22}(\vec{\alpha}, \mathbf{a}) = \{(\alpha_1, \alpha_2) : \alpha_1+\alpha_2 > -1, v_1\alpha_1+v_2\alpha_2 < 1, [1-2v_1]\alpha_1+[1-2v_2]\alpha_2 < 1\}$. As far as the heterogeneity is concerned, three special cases are of interest:

- The traders differ by extrapolation rates $\alpha_1 \neq \alpha_2$ (but with the same $\mathbf{a}_2$-process). That is, $a_{jk} = a_k$ for $j, k = 1, 2$ and hence $v_1 = v_2 \equiv v$ (say) and $v \in [0, 1]$.
- The traders differ by the $\mathbf{a}_2$-process (but with the same extrapolation rate $\alpha_1 = \alpha_2$). That is, $\alpha_1 = \alpha_2 \equiv \alpha_o$ (say).
- The traders differ by both the $\mathbf{a}_2$-process and extrapolation rates ($\alpha_1 \neq \alpha_2$).

A detailed bifurcation analysis is given in Appendix A.7. In all these three cases, along the Neimark-Hopf boundary, the character of bifurcations is defined by the values of $\rho = 2\cos(2\pi w)$. Table 3 summarize the region of ρ for heterogeneous $\mathbf{a}_2$-process. One can see that the different weight vectors affect the variety of bifurcations along the Neimark-Hopf boundary. In particular, when $v_1 \neq v_2$ and $\alpha_1 \neq \alpha_2$, then $\rho \in [-2, 2]$. Therefore, compared to the cases of either $v_1 = v_2$ or $\alpha_1 = \alpha_2$, the heterogeneous $\mathbf{a}_2$ process can generate a wider range of resonance and quasi-periodic orbit bifurcations than the homogeneous $\mathbf{a}_2$ (i.e. either $v_1 = v_2$ or $\alpha_1 = \alpha_2$) process does.

(v_1, v_2)	(α_1, α_2)	Region of ρ
$v_1 = v_2$	$\alpha_1 \neq \alpha_2$	[-2, 0]
$v_1 \neq v_2$	$\alpha_1 = \alpha_2$	[-2, 0]
$v_1 \neq v_2$	$\alpha_1 \neq \alpha_2$	[-2, 2]

TABLE 3. The corresponding region of $\rho = 2\cos(2\pi w)$ for the heterogeneous $\mathbf{a}_2$-process

5. DYNAMICS OF COBWEB MODEL WITH HETEROGENEOUS BELIEFS

As an application of the previous analysis, this section considers an extended version of Brock and Hommes' cobweb model (1997*b*) and investigates the effect on its dynamics of recursive L- and general heterogeneous $\mathbf{a}_L$-processes.

Consider Brock and Hommes' cobweb model (1997*b*) with m groups of agents using different expectations functions H_j $(j = 1, \cdots, m)$. The market equilibrium is described by

$$D(p_t) = \sum_{j=1}^{m} n_{j,t-1} S(H_j(\overrightarrow{P}_{t-1})), \tag{5.1}$$

where D and S are demand and supply functions, $\overrightarrow{P}_{t-1} = (p_{t-1}, p_{t-2}, \cdots, p_{t-L})$ is a vector of past prices and $n_{j,t-1}$ $(j = 1, \cdots, m)$ is the fraction of the jth-group of agents at the beginning of period t (the subscript $t-1$ indicating that this fraction was formed in $[t-1, t)$). H_j is the expectation of the jth-group of agents on the price in period t, which can be obtained at *information cost* $C_j(\geq 0,\ j = 1, \cdots, m)$[8]. As in Brock and Hommes (1997*b*) (see Hommes (1998) also for more details.)

$$n_{j,t} = \exp[-\beta((p_t - H_j(\overrightarrow{P}_{t-1}))^2 + C_j)]/Z_t, \tag{5.2}$$

where $Z_t = \sum_{j=1}^{m} \exp[-\beta((p_t - H_j(\overrightarrow{P}_{t-1}))^2 + C_j)]$ so that $\sum_{j=1}^{m} n_{j,t} = 1$. The parameter β is called the *intensity of choice*, measuring how fast agents switch expectation functions.

[8]In Brock and Hommes' cobweb model (1997*b*), the information is freely available for the agents using the naive expectation and it is not free for the agents using more sophisticated predictions, say rational expectations.

To keep the model simple and to focus on the stability properties affected by heterogeneity in expectations, both demand and supply are assumed to be linear, thus

$$D(p_t) = a - bp_t, \qquad S(H_j(\overrightarrow{P}_{t-1})) = dH_j(\overrightarrow{P}_{t-1}) \tag{5.3}$$

with constants $b, d \geq 0$. Without loss of generality fix $a = 0$ so that the steady-state equilibrium price $p^{eq} = 0$ and all 'prices' are then (positive or negative) deviations from their steady-state equilibrium price. Assume the expectations functions are formed according to the general $\mathbf{a}_L$-process:

$$H_j(\overrightarrow{P}_{t-1}) = g_j \sum_{k=1}^{L_j} a_{jk} p_{t-k} \tag{5.4}$$

with $a_{jk} \geq 0$, $\sum_{k=1}^{L_j} a_{jk} = 1$, $g_j \in \mathbb{R}$ for $j = 1, \cdots, m$ and $k = 1, \cdots, L_j$. In particular, when $g_j = 0$, the j-th group of agents is called *fundamentalists* who 'know' the equilibrium steady-state price $p^{eq} = 0$ and believe that prices will return to the steady state. When $g_j \neq 0$, the j-th group of agents is called *trend traders* (or chartists) who believe that tomorrow's price will be g_j (the extrapolation rate) times the weighted average of the past L_j prices. In particular, when $g_j > 0 (< 0)$, the agents are called *trend followers* (*contrarians*).

Equations (5.1)-(5.4) lead to the following adaptive belief system (see e.g. Brock and Hommes (1997*b*), (1998)):

$$p_t = -\frac{d}{b} \sum_{j=1}^{m} n_{j,t-1} g_j \sum_{k=1}^{L_j} a_{jk} p_{t-k} \tag{5.5}$$

$$n_{j,t} = exp\Big[- \beta \Big([p_t - g_j \sum_{k=1}^{L_j} a_{jk} p_{t-k}]^2 + C_j \Big) \Big] / Z_t \qquad (j = 1, \cdots, m) \tag{5.6}$$

with

$$Z_t = \sum_{j=1}^{m} exp\Big[- \beta \Big([p_t - g_j \sum_{k=1}^{L_j} a_{jk} p_{t-k}]^2 + C_j \Big) \Big].$$

As a simple example of the above heterogeneous model (5.5)-(5.6), Hommes (1998) considers the case of the fundamentalists versus trend traders (following an $AR(1)$

process), that is

$$H_1(\overrightarrow{P}_{t-1}) = 0, \qquad H_2(\overrightarrow{P}_{t-1}) = g_o p_{t-1}.$$

Through this simple example, Hommes illustrates how price expectations affect actual price behaviour. He finds that belief in a strong positive (negative) auto-correlation in prices at the first lag may lead to negative (positive) auto-correlation in actual prices. Hommes goes on to investigate the consistency of the expectations.

Let $L = \max_{1\le j\le m}\{L_j\}$. Then the system (5.5)-(5.6) is an $(L+m)$-dimensional (nonlinear) system with equilibrium $E = (p^{eq}, \cdots, p^{eq}, n_1^{eq}, \cdots, n_m^{eq})$, where $p^{eq} = 0$ and $n_j^{eq} = \exp(-\beta C_j)/\sum_{k=1}^{m}\exp(-\beta C_k)$ $(j = 1, \cdots, m)$. Linearizing the system at the equilibrium E, it is found that the local stability of the equilibrium E is essentially determined by the Lth-order difference equation

$$x_t + \frac{d}{b}\sum_{j=1}^{m} n_j^{eq} g_j \sum_{k=1}^{L_j} a_{jk} x_{t-k} = 0, \tag{5.7}$$

which is the form of (2.6) with $\alpha_j = d g_j n_j^{eq}/b\,(j = 1, \cdots, m)$. Therefore the results of Sections 3 and 4 can be applied.

In the following discussion, consider a model of three groups of agents: contrarians ($g_1 < 0$), trend followers ($g_2 > 0$) and the fundamentalists ($g_3 = 0$). Several different aspects of heterogeneous learning, including

- the aggregated expectations effect;
- the lag length effect of the heterogeneous recursive L-process;
- the general $\mathbf{a}_L$-process effect,

are discussed. It is found that the stability regions, types of bifurcation and routes to complicated price fluctuation of these different aspects affect the dynamics of the nonlinear adaptive model in very different ways.

In the following examples, we choose the set of parameters [9]

$$b = 0.5, d = 1.35, \beta = 5, C_1 = C_2 = 0, C_3 = 1.$$

[9]Here it is assumed that there is a cost of information for the fundamentalists, but it is cost-free for the trend traders.

Then $E = (p^{eq}, n_1^{eq}, n_2^{eq}, n_3^{eq}) = (0, 0.498, 0.498, 0.004)$. Let $\bar{\delta} = b/(dn_o^{eq})$, where $n_o^{eq} \equiv n_1^{eq} = n_2^{eq} = 0.498$. Then $\bar{\delta} = 0.74574$.

Example 1 — *Aggregated Expectations Effect.* To see the effect of the aggregated extrapolation rate on the stability of E, assume both trend traders follow the recursive L-process, then the stability region of $E = (p^{eq}, n_1^{eq}, n_2^{eq}, n_3^{eq})$ is given by

$$-1 < \alpha_1 + \alpha_2 = \frac{d}{b}[n_1^{eq} g_1 + n_2^{eq} g_2] < L. \tag{5.8}$$

For the parameters selected above, condition (5.8) becomes

$$-\bar{\delta} < g \equiv g_1 + g_2 < L\bar{\delta} \qquad \text{with} \qquad \bar{\delta} = \frac{b}{dn_1^{eq}} = 0.74574. \tag{5.9}$$

Let $L = 1$ and consider three special cases.

(i) In the case of the two groups (trend traders versus fundamentalists) model, $p^{eq} = 0$ is LAS when the extrapolation rate (g_1 or g_2) of the trend traders satisfies $-\delta < g_1, g_2 < \delta \equiv \bar{\delta}/2$ (which is called *stable extrapolation rate* for the convenience of the following discussion). In such a case, the aggregated extrapolation rate $g \equiv g_1 + g_2$ of the three groups model satisfies $-\bar{\delta} = -2\delta < g < 2\delta = \bar{\delta}$, implying E is LAS. Therefore, adding a third group of trend traders with stable extrapolation rate to the two groups stable model leads $p^{eq} = 0$ of the corresponding three groups model to be stable.

(ii) Let $g_1 = -1$. Then $p^{eq} = 0$ of the three groups model is LAS for $g_2 \in (0.2568, 1.74)$. However, either $g_1 = -1$ or $g_1 \in (0.37287, 1.74)$ is an unstable extrapolation rate for the two groups model. This indicates that, if the two unstable extrapolation rates of the two trend traders are balanced such that the aggregated extrapolation rate g for the three groups model stays in the stability region, adding the third group to the two group model can stabilize the price dynamics.

(iii) Consider the two groups model with $g_1 \in (-0.37287, 0)$. Then $p^{eq} = 0$ is LAS for the two groups model. Now add a third group of trend followers with a significantly unstable extrapolation rate $g_2 = 2$. Then g stays outside the stability region of the three group model and hence $p^{eq} = 0$ is unstable. This indicates the destabilizing effect of adding the third group when it has a significantly unstable (or divergent) extrapolation rate.

Both (ii) and (iii) demonstrate the *double edged effect* of heterogeneous beliefs.

Example 2 — *Heterogeneous Recursive L Effect.* Assume both trend traders follow recursive L-processes with lag lengths $1 \le L_1, L_2 \le 3$. The reduced homogeneous recursive case is obtained when $L_1 = L_2$. To see the variety of bifurcations, consider the case of $(L_1, L_2) = (1, 2)$. For the discussion of the case $(L_1, L_2) = (2, 3)$, see Appendix A.9.

For $(L_1, L_2) = (1, 2)$, based on the analysis in Section 3, the stability region of $E = (p^{eq}, n_1^{eq}, n_2^{eq}, n_3^{eq})$ is given by

$$D'_{12}(\vec{\alpha}) \equiv \{(\alpha_1, \alpha_2) : -1 < \alpha_1 + \alpha_2, \alpha_1 < 1, \alpha_2 < 2\}, \quad \alpha_j = dg_j n_j^{eq}/b \quad (j = 1, 2)$$

with saddle-node boundary $g_1 + g_2 = -\bar{\delta}$ for $g_1 \in [-3\bar{\delta}, \bar{\delta}]$, flip boundary $g_1 = \bar{\delta}$ for $g_2 \in [-2\bar{\delta}, 2\bar{\delta}]$ and Neimark-Hopf boundary $g_2 = 2\bar{\delta}$ for $g_1 \in [-3\bar{\delta}, \bar{\delta}]$. Let (0.01, -0.02, 0.49, 0.49, 0.02) be the initial value of $(p_t, p_{t-1}, n_{1,t}, n_{2,t}, n_{3,t})$. For $g_1 = -2\bar{\delta}$ and g_2 near $\bar{\delta}$, the fixed equilibrium E is stable for $g_2 > \bar{\delta}$ (say, $g_2 = 0.75$) and unstable for $g_2 < \bar{\delta}$ (say, $g_2 = 0.74$). In fact, numerical simulations show that, for $g_2 \in [0.2, 0.74]$, the fixed equilibrium E is unstable and the solutions converge to fixed values, which are different from E, while for $g_2 = 0.1$, the phase plot shows the system has an attracting closed curve encircling the fixed equilibrium E.

- Along the flip boundary, period doubling bifurcations occur. Let $g_2 = \bar{\delta}$ be fixed and Fig.3(a) shows the phase plot when g_1 is near the flip boundary ($g_1 = \bar{\delta}$). For $g_1 = 0.74 < \bar{\delta}$, the solution converges to E, for $g_1 = 0.8 > \bar{\delta}$, E is unstable and the solution converge to a period 2 cycle, while for $g_1 = 1$, the solution converges to an attractor consisting two separate closed orbits, symmetric about the original point.
- Along the Neimark-Hopf boundary $g_2 = 2\bar{\delta}$, bifurcations for various values of g_1 can be either periodic resonances or quasi-periodic orbits. For example,

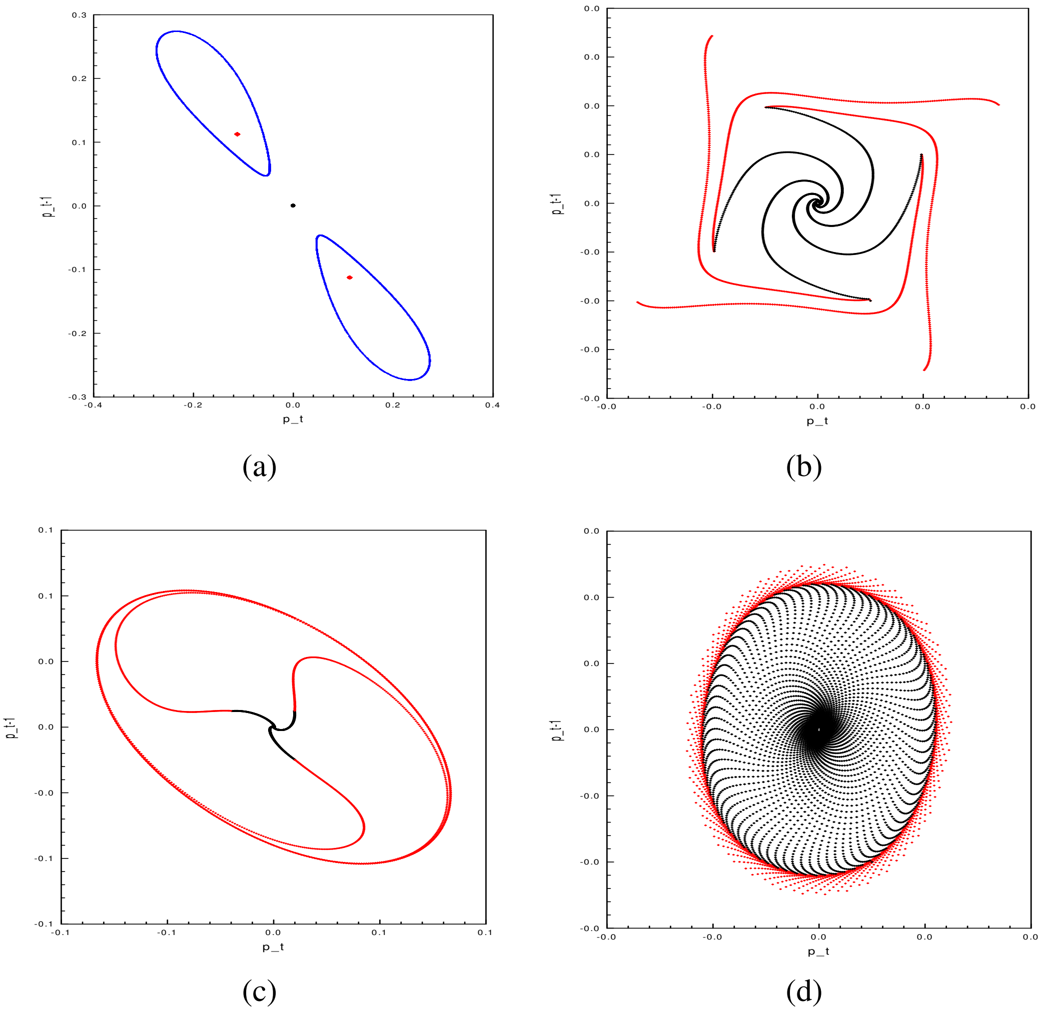


FIGURE 3. Phase plot near the boundaries of the stability region. (a) Phase plot of attractors for $g_2 = \bar{\delta} = 0.74574$ and g_1 near the flip boundary ($g_1 = \bar{\delta}$) — E for $g_1 = 0.74 (< \bar{\delta})$, a period 2 cycle for $g_1 = 0.8$ and two closed orbits for $g_1 = 1$. (b) Phase plot for $g_1 = -\delta$ and $g_2 = 1.48, 1.4845$, near the Neimark-Hopf boundary ($g_2 = 2\bar{\delta}$) with $1:4$ periodic (resonance) bifurcation. (c) Phase plot for $g_1 = -\delta$ and $g_2 = 1.485, 1.5$, near the Neimark-Hopf boundary ($g_2 = 2\bar{\delta}$) with $1:3$ periodic (resonance) bifurcation. (d) Phase plot for $g_1 = 0.8$ and $g_2 = 1.483, 1.4831$, near the Neimark-Hopf boundary ($g_2 = 2\bar{\delta}$) with a different bifurcation.

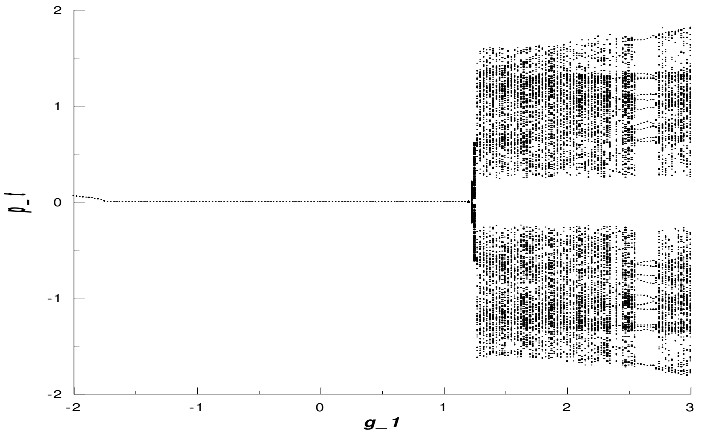
$g_1 = -\bar{\delta}$ corresponds to period 4 bifurcation, while $g_1 = 0$ corresponds to period 3 bifurcation. For $g_1 = -\bar{\delta}$, Fig.3(b) shows the phase plot for g_2 near the 1:4 periodic resonance bifurcation value $g_2 = 2\bar{\delta}$. When $g_2 = 1.48 < 2\bar{\delta}$, the solution converges to E periodically with period 4, while when $g_2 = 1.4845$ the solution diverges periodically with period 4. For $g_1 = 0$, Fig.3(c) shows the phase plot for g_2 near the 1:3 periodic resonance bifurcation value $g_2 = 2\bar{\delta}$.

For $g_2 = 1.485$, the solution converges to E periodically with period 3, while for $g_2 = 1.5$, through a 1:3 resonance bifurcation, the solution tends to an attracting closed curve encircling the fixed equilibrium E. Fig.3(d) shows a different type of bifurcation near the Neimark-Hopf boundary, where $g_1 = 0.8$ is fixed. Numerical simulations show that there exists a closed orbit for some $g_2 \in (1.483, 1.4831)$ such that the solution converges to the fixed equilibrium E for $g_2 = 1.483$ and diverges from the closed orbit for $g_2 = 1.4831$.

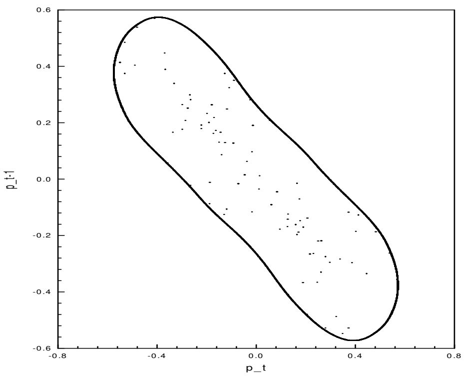
Example 3 — *Heterogeneous* $\mathbf{a}_2$*-Process Effect.* To illustrate the effect of the heterogeneous $\mathbf{a}_L$ process on the dynamics of the cobweb model, assume expectations of the trend traders follow an $\mathbf{a}_2$-process. Then the stability region is defined by $D'_{22}(\vec{\alpha}, \mathbf{a})$. Select the parameters as before and consider two different cases.

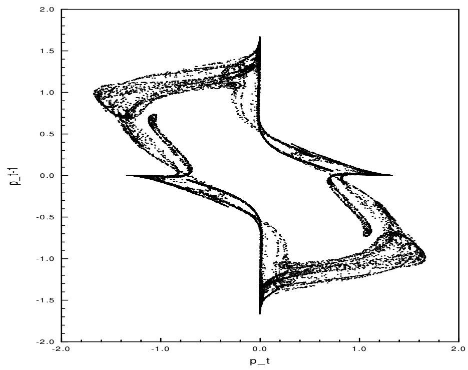
Case (1). Assume both groups of trend traders follow the arithmetic weights process with $L_1 = L_2 = 2$, i.e., $(a_{i1}, a_{i2}) = (2/3, 1/3)$ for $j = 1, 2$. Then the stability region (in terms of the extrapolation rates (g_1, g_2)) is given by (note that $n_1^{eq} = n_2^{eq}$) $-1 < \frac{d}{b} n_2^{eq}[g_1 + g_2] < 3$, that is, $-0.7432 = -\bar{\delta} < g_1 + g_2 < 3\bar{\delta} = 2.2297$. Also, the saddle-node boundary is $g_1+g_2 = -\bar{\delta}$ and the Neimark-Hopf boundary is $g_1+g_2 = 3\bar{\delta}$. In fact, the only type of bifurcation along the Neimark-Hopf boundary is a period two bifurcation. For a given $g_2 = 1$, Fig.4 (a) shows the bifurcation diagram for the parameter $g_1 \in (-2, 3)$. It shows that, when g_1 crosses the saddle-node boundary value $g_1 = -1.7432$, E becomes unstable and solutions converge to a non zero solution. However, when g_1 crosses the Neimark-Hopf boundary value $g_1 = 1.2297$, through a period 2 bifurcation, the behavior of the solution become complicated. Now, for fixed $g_1 = 2.0$, Fig. 4(b) shows a phase plot of (p_t, p_{t-1}) with $g_2 = 0.5$ (so that $g_1+g_2 = 2.5$ is outside of the local stability region), which indicates the instability of the equilibrium E and the solution tends to form a closed orbit as t increases. Increasing g_2 further to $g_2 = 1.0$ leads the solution to a chaotic price fluctuation, as shown in Fig.4 (c) for the phase plot of (p_t, p_{t-1}).

Case (2). Next assume the first group of trend traders follows the general $\mathbf{a}_2$-process and the second group of trend traders follows the recursive 2-process. That



(a)





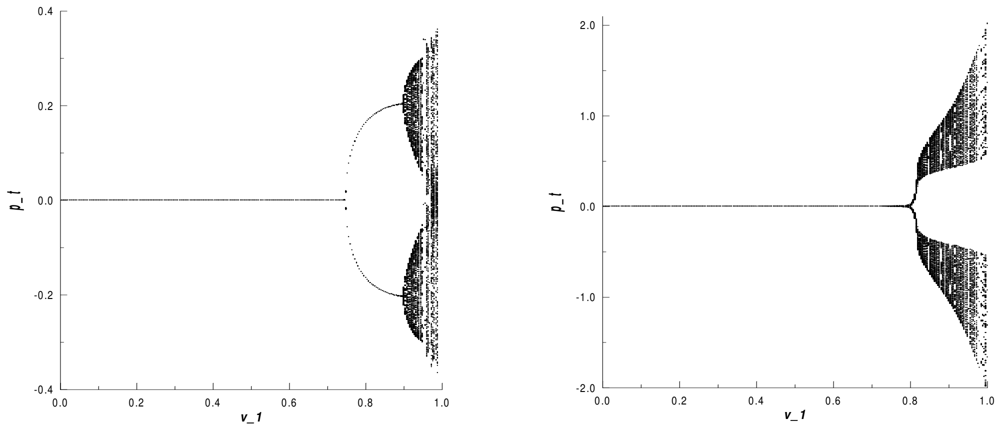
(b) (c)

FIGURE 4. (a) Bifurcation diagram for $g_2 \in [-2, 3]$ of the three groups model with $g_1 = 1$, $g_3 = 0$ and $\mathbf{a}_2 = (2/3, 1/3)$; (b) Phase plot with $(g_1, g_2, g_3) = (2.0, 0.5, 0)$; (c) Phase plot with $(g_1, g_2, g_3) = (2.0, 1.0, 0)$

is, $\mathbf{a_1} = (1 - v_1, v_1)$ and $\mathbf{a_2} = (1/2, 1/2)$. Then the stability region of E is bounded by the saddle-node boundary $g_1 + g_2 = -\bar{\delta}$, the flip boundary $(1 - 2v_1)g_1 = \bar{\delta}$ and the Neimark-Hopf boundary $v_1 g_1 + g_2/2 = 1$. Different types of bifurcations can be generated along those boundaries. For given extrapolation rates (g_1, g_2), the bifurcation dynamics of the parameter v_1 is shown in Fig. 5. In Fig. 5(a), $(g_1, g_2) = (-1.5, 1)$, while in Fig. 5 (b), $(g_1, g_2) = (0.3, 1)$. It is observed that:

- An increase (decrease) of the weight ($a_{11} = v_1$) to p_{t-1} (rather than the weight ($a_{12} = 1 - v_1$) to p_{t-2}) stabilizes the price dynamics. In contrast, a decrease of the weight to p_{t-1} leads the equilibrium $p^{eq} = 0$ to be unstable, resulting in complicated price fluctuations;

- For the two trend traders groups, the stability region, in terms of v_1, of the equilibrium $p^{eq} = 0$, becomes larger when their extrapolation rates are balanced ($g_1 g_2 < 0$, as shown in Fig. 5(a)) than when they are unbalanced ($g_1 g_2 > 0$, Fig. 5(b)).



(a) (b)

FIGURE 5. Bifurcation diagram of v_1 with $\mathbf{a_1} = (1 - v_1, v_1)$, $\mathbf{a_2} = (1/2, 1/2)$ and $(g_1, g_2) = (-1.5, 1)$ for (a) and $(g_1, g_2) = (0.3, 1)$ for (b).

6. CONCLUSION

This paper generalize the study on the dynamics of homogeneous expectations and learning in Chiarella and He (2001a) to the case of heterogeneous expectations and learning via recursive L-processes and general $\mathbf{a}_L$-processes. It shows how the local stability of the equilibrium is affected by the recursive L-process and $\mathbf{a}_L$-process, the various types of bifurcation that can arise and, in the case of the cobweb model, routes to complicated dynamics. Our results show that heterogeneous expectations and learning can lead to very rich dynamics and our results might be summarized as follows:

- When agents use homogeneous recursive L-processes, the stability region is completely characterized by the system parameters and L. Along the boundaries of the stability region, the only types of bifurcation that can be generated are either saddle-node or $1 : (L + 1)$ resonance. However, when the agents use heterogeneous recursive L-processes, the stability region is bounded by saddle-node, flip and Neimark-Hopf boundaries. Along the flip boundary, the system has period doubling bifurcations. Along the Neimark-Hopf boundary, the type of the bifurcation is characterized by the complex eigenvalues $\exp(\pm 2\pi w i)$, which in turn is determined by $\rho = 2\cos(2\pi w)$. The region of ρ depends essentially upon two parameters: the lag lengths used in learning processes and the aggregated extrapolation rate. In particular, different lag lengths can generate different types of resonance and quasi-periodic orbits, leading to different routes to complicated fluctuations.

- When agents use general heterogeneous $\mathbf{a}_L$-processes, the stability region and bifurcation dynamics become more complicated. Apart from the two parameters for the heterogeneous recursive process — the lag length used in the learning process and the aggregated extrapolation rate, the weighting $\mathbf{a}_L$-process also plays an important role. The three examples in Section 5 on the adaptive cobweb model demonstrate the effect of the lag lengths, aggregated extrapolation rate and weight process on the price dynamics and they show various stability regions, types of bifurcation and routes to complicated price fluctuations.

The expectations functions considered in this paper are some of the simplest learning processes in which all the weights on the past states are constants. The analysis has shown how they yield very rich dynamics in terms of the stability, bifurcation and routes to complicated dynamics. In practice, agents revise their expectations by adapting the weights in accordance to observations. How the learning affects the dynamics in this case is a question left for future research.

APPENDIX A

A.1. **Rouche's Theorem.** (see Elaydi (1996)) If the complex functions $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve γ, and if $|g(z)| < |f(z)|$ on γ, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside γ.

A.2. **Proof of Proposition 3.2.** When $m = 2$ and $1 \leq L_1 < L_2$, the characteristic polynomial (2.7) has the form

$$\Gamma(z) = z^{L_2} + \left(\frac{\alpha_1}{L_1} + \frac{\alpha_2}{L_2}\right)[z^{L_2-1} + \cdots + z^{L_2-L_1}] + \frac{\alpha_2}{L_2}[z^{L_2-L_1-1} + \cdots + z + 1].$$

Let $f(z) = z^{L_2}$ and $g(z) = \Gamma(z) - f(z)$. Then on $|z| = 1$, $|f(z)| = 1$ and $|g(z)| \leq L_1\left|\frac{\alpha_1}{L_1} + \frac{\alpha_2}{L_2}\right| + (L_2 - L_1)\frac{|\alpha_2|}{L_2}$. Thus, under the condition (3.4), $|g(z)| < |f(z)|$ on $|z| = 1$. Note that $f(z)$ has L_2 zeros inside $|z| = 1$. It follows from Rouche's Theorem that both $f(z)$ and $\Gamma(z) = f(z) + g(z)$ have same number of zeros inside $|z| = 1$, which implies that all the eigenvalues of $\Gamma(z)$ lie inside $|z| = 1$. Therefore, x^* is LAS. *Q.E.D*

A.3. **Lemma.** The following lemma is a combination of Jury's test (see pp.180-181 in Elaydi (1996)) and bifurcation analysis in Sonis (2000).

Lemma 2.

- *All the eigenvalues λ of the characteristic polynomial $\lambda^2 + b_1\lambda + b_2 = 0$ satisfy $|\lambda| < 1$ iff*

$$-1 < b_2 < 1, \qquad |b_1| < 1 + b_2. \tag{A.1}$$

  Let $D(b_1, b_2)$ be the region in (b_1, b_2) space defined by (A.1). Then, $\lambda_{1,2} \in \mathbb{C}$ satisfying $|\lambda_{1,2}| = 1$ lie along the boundary $b_2 = 1$, one of the eigenvalues $\lambda = -1$ lies along the boundary $b_1 = 1 + b_2$, and one of the eigenvalues $\lambda = 1$ lies along the boundary $b_1 = -(1 + b_2)$.

- *All the eigenvalues λ of the characteristic polynomial $\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0$ satisfy $|\lambda| < 1$ iff $\Pi_j > 0$ and $c_2 < 3$,*[10] *where*

$$\Pi_1 \equiv 1 + c_1 + c_2 + c_3, \quad \Pi_2 \equiv 1 - c_1 + c_2 - c_3, \quad \Pi_3 \equiv 1 - c_2 + c_1c_3 - c_3^2.$$

  Furthermore, on $\Pi_1 = 0$ at least one of the eigenvalues is equal to 1; On $\Pi_2 = 0$ at least one of the eigenvalues is equal to -1 and on $\Pi_3 = 0$, the three eigenvalues satisfy $\lambda_{1,2} \in \mathbb{C}$ and $\lambda_3 \in \mathbb{R}$ with $|\lambda_{1,2}| = 1$ and $\lambda_3 \in [-1, 1]$.

[10]The condition $c_2 < 3$ should be added in the corresponding results in Sonis (2000). We are indebted to Laura Gardini for drawing this to our attention.

A.4. **Table of values for resonance.** Table 4 lists $p:q$ resonances and corresponding values of $\rho = 2\cos(2\pi p/q)$ for $q < 12$, It can be found in Sonis (2000) and is included here for convenience.

q	p	$\rho = 2\cos(2\pi p/q)$		q	p	$\rho = 2\cos(2\pi p/q)$
2	1	-2		9	2,7	0.34730
3	1	-1			4,5	-1.87939
4	1	0		10	1,9	1.61803
5	1,4	0.61803			3,7	-0.61803
	2, 3	-1.61803		11	1,10	1.68251
6	1,5	1			2,9	0.83083
7	1,6	1.24698			3,8	-0.28363
	2,5	-0.44504			4,7	-1.30972
	3,4	-1.80194			5,6	-1.91899
8	1,7	1.41421		12	1,11	1.73205
	3,5	-1.41421			5,7	-1.73205
9	1,8	1.53209				

TABLE 4. $p:q$ resonance and the corresponding values $\rho = 2\cos(2\pi p/q)$

A.5. **Stability and Bifurcation Analysis of Heterogeneous L-Processes with** $(L_1, L_2) = (1,3)$ **and** $(2,3)$**.** The corresponding characteristic polynomials $\Gamma_{L_1,L_2}(\lambda)$ are

$$\Gamma_{1,3}(\lambda) \equiv \lambda^3 + (\alpha_1 + \alpha_2/3)\lambda^2 + (\alpha_2/3)\lambda + \alpha_2/3 = 0,$$

$$\Gamma_{2,3}(\lambda) \equiv \lambda^3 + (\alpha_1/2 + \alpha_2/3)\lambda^2 + (\alpha_1/2 + \alpha_2/3)\lambda + \alpha_2/3 = 0.$$

Applying Proposition 3.1 and Lemma 2 in Appendix A.3, one can obtain the stability region, respectively

$$\begin{aligned} D'_{13}(\vec{\alpha}) = &\quad \{(\alpha_1, \alpha_2) : -1 < \alpha_1 + \alpha_2, \quad \alpha_1 + \alpha_2/3 < 1, \quad \alpha_2(1-\alpha_1) < 3\}; \\ D'_{23}(\vec{\alpha}) = &\quad \{(\alpha_1, \alpha_2) : -1 < \alpha_1 + \alpha_2, \quad \alpha_2 < 3, \quad (\alpha_1/2 - 1)(\alpha_2/3 - 1) < 1\}. \end{aligned}$$

- In the first case $(L_1, L_2) = (1,3)$, the local stability region $D'_{13}(\vec{\alpha})$ is plotted in Fig.2(a) with flip, saddle-node and Neimark-Hopf boundaries as indicated. Along the Neimark-Hopf boundary, $\lambda_{1,2} = \cos(2\pi w) \pm i\sin(2\pi w)$, $\lambda_3 \in [-1,1]$ for $w \in \mathbb{R}$. Let $\rho = 2\cos(2\pi w)$, then $\alpha_1 = -\rho$, $\alpha_2 = 3/(1-\rho)$ and $\rho \in [0,2]$. The types of resonance bifurcation are different from the case of $L_1 = 1$ and $L_2 = 2$, in which $\rho \in [-2,2]$, indicating a wide range of resonance and quasi-periodic orbits bifurcations.

- In the case of $(L_1, L_2) = (2, 3)$, the local stability region $D'_{23}(\vec{\alpha})$ is plotted in Fig.2(b) with flip, saddle-node and Neimark-Hopf boundaries as indicated. Different from the previous cases, the Neimark-Hopf boundary constitutes by $(F1) : \alpha_1 = 2, \alpha_2 \in [-3, 3]$ and $(F2) : \alpha_2 = 3, \alpha_1 \in [-4, 2]$.
  - Along $(F1)$, $\lambda_{1,2} = \cos(2\pi/3) \pm i \sin(2\pi/3)$ and $\lambda_3 = -\alpha_2/3 \in [-1, 1]$, hence $1 : 3$ periodic resonance is the only type of bifurcation, there is no quasi-periodic orbit.
  - Along $(F2)$, $\lambda_{1,2} = \cos(2w\pi) \pm i \sin(2w\pi)$ and $\lambda_3 = -1$ for $w \in \mathbb{R}$. $\lambda_3 = -1$ corresponds to a flip (or period doubling) bifurcation, while $\lambda_{1,2}$ determines the types of bifurcations.

  Note that, along the Neimark-Hopf boundary $(F2)$, $\rho = 2\cos(2\pi w) = -\alpha_1/2$ and hence $\rho \in [-1, 2]$. For $\rho \in [-1, 2]$, irrational w correspond to quasi-periodic orbits, while rational w corresponds to periodic resonances. Checking with Table 4 for $\rho \in [-1, 2]$, one can see the existence of various $p : q$ periodic resonances. For example, $\alpha_1 = 2$ corresponds to $1 : 3$ periodic resonance and $\alpha_1 = 0$ corresponds to period 4 cycle starting the Feigenbaum period doubling route to chaos. Since each point along the Neimark-Hopf boundary $\alpha_2 = 3, \alpha_1 \in [-4, 2]$ also corresponds to a flip type of bifurcation, theoretical analysis of such types of bifurcation can be exceedingly complicated and is not yet completely understood.

A.6. **Proof of Proposition 4.1.** Let $f_1(z) = z^L$ and $g_1(z) = \Gamma(z) - f_1(z)$. Then, under the condition (i), $|g_1(z)| < |f_1(z)|$ on $|z| = 1$ and thus LAS follows from Rouche's Theorem. To prove the second part, let $f(z) = \Big(\sum_{j=1}^m \alpha_j a_{jk_o}\Big) z^{L-k_o}$ and $g(z) = \Gamma(z) - f(z)$. Then, under the condition (ii), $|g(z)| < |f(z)|$ on $|z| = 1$. By Rouche's theorem, both $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside $|z| = 1$. Note that $f(z)$ has $L - k_o$ zeros inside $|z| = 1$ and thus there exists at least one eigenvalue z of $f(z) + g(z)$ satisfying $|z| \geq 1$. Therefore x^* is unstable. *Q.E.D*

A.7. **Bifurcation Analysis of the Heterogeneous $\mathbf{a}_2$-Process. Case 1**: In this case $a_{jk} = a_k$ for $j, k = 1, 2$. Hence $v_1 = v_2 \equiv v$ (say) and $v \in [0, 1]$. Correspondingly, $D'_{22}(\alpha, \mathbf{a}) = \{(\alpha_1, \alpha_2) : -1 < \alpha_1 + \alpha_2, v(\alpha_1 + \alpha_2) < 1, [1 - 2v](\alpha_1 + \alpha_2) < 1\}$.

Let $\alpha = \alpha_1 + \alpha_2$ and replace α and v_2 by $\alpha_1 + \alpha_2$ and v respectively, then $D'_{22}(\vec{\alpha}, \mathbf{a})$ defines the same stability region as $D_2(\vec{\alpha}, \mathbf{a})$ for the homogeneous recursive 2-process in Chiarella and He (2001*a*). In particular, $v = 1/2$ leads to the stability region of the heterogeneous recursive 2-process while $v = 1/3$ leads to the largest stability region: $-1 < \alpha_1 + \alpha_2 < 3$. Also, $D'_{22}(\vec{\alpha}, \mathbf{a}) \subset D'_{22}(\vec{\alpha}) = \{\vec{\alpha}; -1 < \alpha < 2\}$ for $v \notin (1/4, 1/2)$ while $D'_{22}(\vec{\alpha}) \subset D'_{22}(\vec{\alpha}, \mathbf{a})$ for $v \in [1/4, 1/2]$. Along the Neimark-Hopf boundary $\alpha = 1/v$ for $v \in [1/3, 1]$, bifurcation is characterized by the eigenvalues $\lambda_{1,2} = \exp(\pm 2\pi w i)$ with w satisfying $\rho \equiv 2\cos(2\pi w) = 1 - 1/v \in [-2, 0]$. See Chiarella and He (2001*a*) for more detailed discussion.

Case 2: Now assume $\alpha_1 = \alpha_2 \equiv \alpha_o$ (say). Then $D'_{22}(\vec{\alpha}) = \{\alpha_o : -1/2 < \alpha_o < 1\}$ and $D'_{22}(\vec{\alpha}, \mathbf{a}) = \{\alpha_o : \alpha_o > -1/2, v\alpha_o < 1, (1 - v)\alpha < 1/2\}$ with $v = v_1 + v_2 \in [0, 2]$. Also, $D'_{22}(\vec{\alpha}, \mathbf{a}) \subset D'_{22}(\vec{\alpha})$ for $v \notin (1/2, 1)$, $D'_{22}(\vec{\alpha}) \subset D'_{22}(\vec{\alpha}, \mathbf{a})$ for $v \in [1/2, 1]$, while $v = v_1 + v_2 = 2/3$ leads to the largest stability region: $-1/2 < \alpha_o < 3/2$[11]. The character of bifurcations along the Neimark-Hopf boundary $v\alpha_o = 1$ is defined by the value of ρ. Along the Neimark-Hopf boundary, $\rho = 1 - 2\alpha_o$ and $v\alpha_o = 1$. It follows from $v \in [2/3, 2]$ that $\rho \in [-2, 0]$.

Case 3: The next example indicates a wide range of stability regions when there is heterogeneity in both $\mathbf{a}_2$ and (α_1, α_2). To illustrate the stability region, for the sake of simplicity, select $v_2 = 1/2$, that is, the first expectation follows a general $\mathbf{a}_2$-process but the second one follows the recursive 2-process. Then $D'_{22}(\vec{\alpha}, \mathbf{a}) = \{(\alpha_1, \alpha_2) : -1 < \alpha_1 + \alpha_2, (1 - 2v_1)\alpha_1 < 1, v_1\alpha_1 + \alpha_2/2 < 1\}$. For different values of $v_1 \in [0, 1]$, the stability region $D'_{22}(\vec{\alpha}, \mathbf{a})$ is plotted in Fig.6. One can see that $D'_{22}(\vec{\alpha}, \mathbf{a})$ is a bounded triangular region, except the case $v_1 = 1/2$ in which it becomes an unbounded strip $D_{22}(\vec{\alpha}) = \{(\alpha_1, \alpha_2) : -1 < \alpha_1 + \alpha_2 < 2\}$.

In general, for $v_1 \neq v_2$, the stability region is bounded by the saddle-node curve $\alpha_1 + \alpha_2 = -1$, flip curve $(1 - 2v_1)\alpha_1 + (1 - 2v_2)\alpha_2 = 1$ and the Neimark-Hopf curve $v_1\alpha_1 + v_2\alpha_2 = 1$. In particular, on the Neimark-Hopf boundary, the character of bifurcations is defined by the eigenvalues $\lambda_{1,2} = \cos(2\pi w) \pm i\sin(2\pi w)$ with values of w satisfying $\rho \equiv 2\cos(2\pi w) \in [-2, 2]$[12]. Also, along the Neimark-Hopf boundary, $\alpha_1 + \alpha_2 = 1 - \rho$. Hence, for given

[11] In this case, to obtain the largest stability region through the $\mathbf{a}_2$-process, one can select (a_{j1}, a_{j2}) either, to be $(2/3, 1/3)$ for $j = 1, 2$ when α_1 and α_2 are not necessarily the same or, to satisfy $a_{12} + a_{22} = 2/3$ when α_1 and α_2 are the same.

[12] See Appendix A.8 for the details.

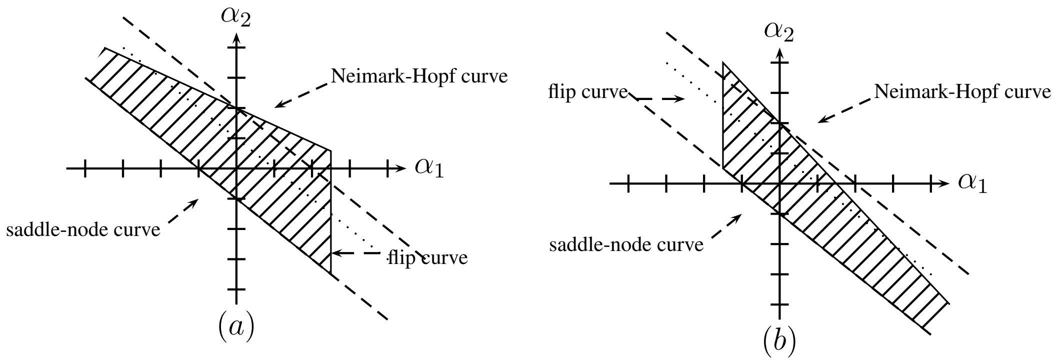


FIGURE 6. Local stability regions of the heterogeneous $\mathbf{a}_2$-process D'_{22} for (a): $0 \le v_1 \le 1/2$; (b): $1/2 < v_1 \le 1$ with saddle-node boundary $\alpha_1 + \alpha_2 = -1$, flip boundary $(1 - 2v_1)\alpha_1 = 1$ and Neimark-Hopf boundary $v_1\alpha_1 + \alpha_2/2 = 1$.

$\rho \in [-2, 2]$, the weights v_1, v_2 satisfy

$$v_1\alpha_1 + v_2(1 - \rho - \alpha_1) = 1. \tag{A.2}$$

For given α_1, equation (A.2) defines the weights (v_1, v_2) which give the same type of bifurcation, defined by ρ (see Appendix A.8). Fig.7(a) illustrates such a situation for fixed $\alpha_1 = 2$ and $\rho = -2, -1, 0, 1, 2$. On the other hand, if one of the $\mathbf{a}_2$-processes is given, equation (A.2) gives a relation between α_1 and the other $\mathbf{a}_2$-process, which corresponds to the same type of bifurcation, defined by ρ. Fig.7(b) illustrates such a situation for fixed $v_2 = 1/2$ and $\rho = -2, -1, 0, 1, 2$. After all, for a given ρ, the surface defined by (A.2) in the three dimensional space (v_1, v_2, α_1) corresponds to the same type of bifurcation. When $w = p/q$ is rational, the system has a $p : q$ periodic resonance bifurcation, while for irrational w, it bifurcates to a quasi-periodic orbit.

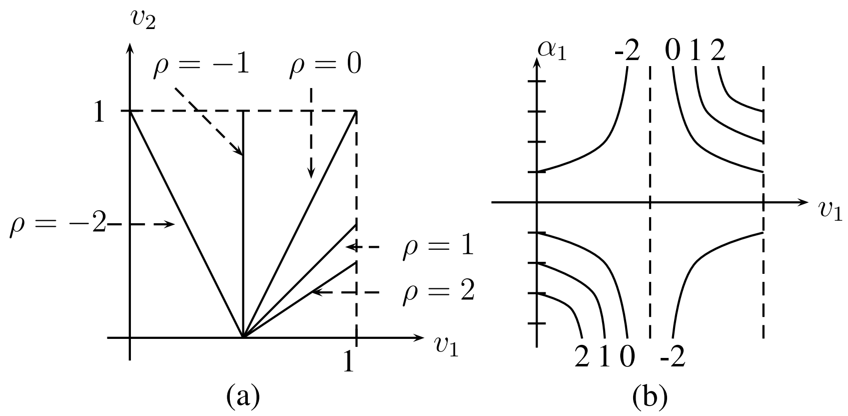


FIGURE 7. Bifurcation curves for $\rho = 0, \pm 1, \pm 2$ on the (v_1, v_2) plane (a) and on the (v_1, α_1) plane (b)

A.8. **Bifurcation along the Neimark-Hopf Boundary.** On the Neimark-Hopf curve, $\lambda_{1,2} = \cos(2\pi w) \pm i\sin(2\pi w)$ for $w \in \mathbb{R}$. It follows from $\lambda_1 + \lambda_2 = -c_1 = -[(1-v_1)\alpha_1 + (1-v_2)\alpha_2]$ and $\lambda_1\lambda_2 = c_2 = v_1\alpha_1 + v_2\alpha_2$ that $\alpha_1 + \alpha_2 = 1 - \rho, v_1\alpha_1 + v_2\alpha_2 = 1$, where $\rho = 2\cos(2\pi w)$. For $v_1 \neq v_2$, solving α_1, α_2 leads to $\alpha_1 = \frac{v_2(1-\rho)-1}{v_2-v_1}, \alpha_2 = \frac{1-v_1(1-\rho)}{v_2-v_1}$. At the same time, solving the intersection of the Neimark-Hopf and flip boundaries leads to $\rho = -2$ and solving the intersection of the Neimark-Hopf and saddle-node boundaries leads to $\rho = 2$. Therefore, along the Neimark-Hopf curve, the eigenvalues $\lambda_{1,2}$ take values w satisfying $\rho = 2\cos(2\pi w) \in [-2, 2]$. Therefore, if $w = p/q$, there exist $p : q$ resonance bifurcations. If w is irrational, quasi-periodic orbits occur. *Q.E.D*

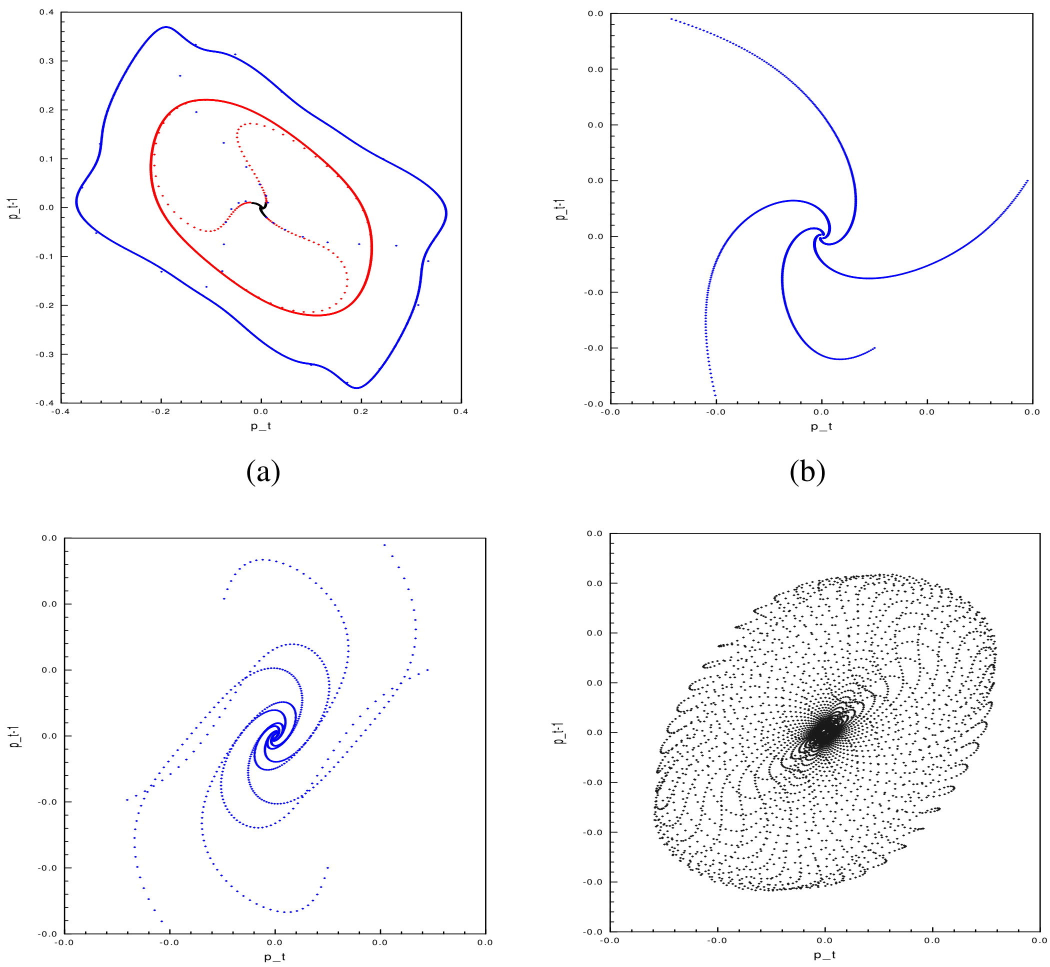
A.9. **Heterogeneous Effect —**$(L_1, L_2) = (2, 3)$**.** Based on the analysis in Section 3, the stability region $D'_{23}(\vec{\alpha})$ of $E = (p^{eq}, n_1^{eq}, n_2^{eq}, n_3^{eq})$ is bounded by the saddle-node boundary $g_1 + g_2 = -\bar{\delta}$ for $g_1 \in [-4\bar{\delta}, 2\bar{\delta}]$, the flip boundary $g_2 = 3\bar{\delta}$ for $g_1 \in [-4\bar{\delta}, 2\bar{\delta}]$ and the Neimark-Hopf boundaries $g_1 = 2\bar{\delta}$ for $g_2 \in [-3\bar{\delta}, 3\bar{\delta}]$ and $g_2 = 3\bar{\delta}$ for $g_1 \in [-4\bar{\delta}, 2\bar{\delta}]$. In this case, both $g_1 = 2\bar{\delta}$ and $g_2 = 3\bar{\delta}$ are the Neimark-Hopf boundary. Along $g_1 = 2\bar{\delta}$, the system has a 1:3 resonance bifurcation. Fig.8(a) shows the phase plot near this Neimark-Hopf boundary. For fixed $g_2 = 0$, the solution converges to E for $g_1 = 1.4 < 2\bar{\delta}$, while for $g_1 = 1.6, 2 > 2\bar{\delta}$, the solution tends to attracting closed curves encircling the fixed equilibrium E.

Along the Neimark-Hopf boundary $g_2 = 3\bar{\delta}$, the system bifurcates all types of resonances and quasi-periodic orbits. Note that this boundary is also the flip boundary. Figs.8(b)-(d) show the phase plot of the solution near this flip-Neimark-Hopf boundary with $(g_1, g_2) = (0, 2.21), (-2\bar{\delta}, 2.2)$ and (-1.56, 2.218), respectively. Time series plots show that all the solutions are symmetric about $p_t = 0$. This symmetry reflects the character of the flip type of bifurcation near the boundary.

References

Balasko, Y. and Royer, D. (1996), 'Stability of competitive equilibrium with respect to recursive and learning processes', *Journal of Economic Theory* **68**, 319–348.

Bray, M. (1983), *Convergence to rational expectations equilibria*, Cambridge Univ. Press, Cambridge, UK. in *Individual Forecasting and Aggregate Outcomes*, Eds. R. Frydman and E.S. Phelps.



(c) (d)

FIGURE 8. Phase plot of the solutions near the Neimark-Hopf boundaries: (a) $g_1 = 2\bar{\delta}$ and $g_1 = 1.4, 1.6, 2$; (b) $(g_1, g_2) = (0, 2.21)$ near a $1 : 4$ resonance bifurcation; (c)$(g_1, g_2) = (-2\bar{\delta}, 2.2)$ near a $1 : 6$ resonance bifurcation and (d) $(g_1, g_2) = (-1.56, 2.218)$ near some (quasi-) periodic resonance bifurcation.

Brock, W. and Hommes, C. (1997*a*), *Models of Complexity in Economics and Finance*, Wiley, New York, chapter 1, pp. 3–44. in *Systems Dynamics in Economic and Finance Models*, Eds. Heij, C., Schumacher, J.M., Hanzon, B., Praagman, C.

Brock, W. and Hommes, C. (1997*b*), 'A rational route to randomness', *Econometrica* **65**, 1059–1095.

Brock, W. and Hommes, C. (1998), 'Heterogeneous beliefs and routes to chaos in a simple asset pricing model', *Journal of Economic Dynamics and Control* **22**, 1235–1274.

Bullard, J. (1994), 'Learning equilibria', *Journal of Economic Theory* **64**, 468–485.

Bullard, J. and Duffy, J. (1999), 'Using Genetic Algorithms to Model the Evolution of Heterogeneous Beliefs', *Computational Economics* **13**, 41–60.

Chiarella, C. and He, X. (2000), *The Dynamics of the Cobweb when Producers are Risk Averse Learners*, Physica-Verlag, pp. 86–100. in *Optimization, Dynamics, and Economic Analysis*, E.J. Dockner, R.F. Hartl, M. Luptacik and G. Sorger (Eds).

Chiarella, C. and He, X. (2001*a*), 'Dynamics of beliefs and learning under $\mathbf{a}_l$-processes – the homogeneous case', *School of Finance and Economics, University of Techonology Sydney* . Working Paper No. 53.

Chiarella, C. and He, X. (2001*b*), 'Heterogeneous beliefs, risk and learning in a simple asset pricing model', *Computational Economics* . Forthcoming.

Day, R. and Huang, W. (1990), 'Bulls, bears and market sheep', *Journal of Economic Behavior and Organization* **14**, 299–329.

Elaydi, S. (1996), *An Introduction to Difference Equations*, Springer, New York.

Evans, G. and Honkapohja, S. (1999), *Learning Dynamics*, chapter 7, pp. 449–542. in *Handbook of Macroeconomics*, Eds. J.B. Taylor and M. Woodford.

Evans, G. and Ramey, G. (1992), 'Expectation calculation and macroeconomics dynamics', *American Economic Review* **82**, 207–224.

Franke, R. and Nesemann, T. (1999), 'Two destabilizing strategies may be jointly stabilizing', *Journal of Economics* **69**, 1–18.

Franke, R. and Sethi, R. (1998), 'Cautious trend-seeking and complex asset price dynamics', *Research in Economics* **52**, 61–79.

Grandmont, J.-M. (1998), 'Expectations formation and stability of large socioeconomic systems', *Econometrica* **66**(4), 741–781.

Hale, J. and Kocak, H. (1991), *Dynamics and bifurcations*, Vol. 3 of *Texts in Applied Mathematics*, Springer-Verlag, New York.

Hommes, C. (1998), 'On the consistency of backward-looking expectations: The case of the cobweb', *Journal of Economic Behavior and Organization* **33**, 333–362.

Kuznetsov, Y. (1995), *Elements of applied bifurcation theory*, Vol. 112 of *Applied mathematical sciences*, SV, New York.

Levy, M. and Levy, H. (1996), 'The danger of assuming homogeneous expectations', *Financial Analysts Journal* **52**(3), 65–70.

Lucas, R. (1978), 'Asset prices in an exchange economy', *Econometrica* **46**, 1229–1145.

Lux, T. (1995), 'Herd behaviour, bubbles and crashes', *Economic Journal* **105**, 881–896.

Lux, T. (1997), 'Time variation of second moments from a noise trader/infection model', *Journal of Economic Dynamics and Control* **22**, 1–38.

Lux, T. (1998), 'The socio-economic dynamics of speculative markets: Interacting agents, chaos, and the fat tails of return distributions', *Journal of Economic Behavior and Organization* **33**, 143–165.

Marcet, A. and Sargent, T. (1989), 'Convergence of least squares learning mechanisms in self referential linear stochastic models', *Journal of Economic Theory* **48**, 337–368.

Sonis, M. (2000), 'Critical bifurcation surfaces of 3d discrete dynamics', *Discrete Dynamics in Nature and Society* **4**, 333–343.